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RESEARCH ARTICLE

Connected closed monophonic number of graphs

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Abstract

In this paper the concept of connected closed monophonic number of a graph is introduced. Let G be a connected graph and $M \subset V(G)$. M is called closed monophonic cover of G if $M = \{v_1, v_2, v_3 \ldots v_k\}$ is obtained by choosing the vertices on the following way. (i) $v_1 \neq v_2$. (ii) $v_i \notin I[M_{i-1}]$ for $3 \leq i \leq k$ and (iii) $I[M_k] = V(G)$ where $M_i = \{v_1, v_2, v_3 \ldots v_i\}$ for all $i = 1, 2, 3 \ldots k$. A set of vertices of a graph is connected closed monophonic cover, if it is closed monophonic cover and the induced sub graph is connected. Connected closed monophonic number of some general graphs such as P_n, C_n, complete graph K_p, bipartite graph $K_{m,n}$ and non-trivial tree are derived.

Keywords

Monophonic cover
Closed monophonic cover
Minimal monophonic set
Connected closed
Monophonic number

Introduction

By a graph $G = (V, E)$ we consider a finite undirected graph without loops or multiple edges. The order and size of G are denoted by m and n respectively. For basic graph theoretic notations and terminology we refer to Buckley and Harary [3]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called $u - v$ *geodesic*. A chord of a path P: $u_1, u_2 \ldots u_n$ is an edge $u_i u_j$ with $j \geq i + 2$. A $u - v$ path is *monophonic path* if it is chord less path. A *monophonic set* of G is a set $M \subset V(G)$ such that every vertex of G is contained in a monophonic path of some pair of vertices of M.The *monophonic number* of a graph G is explained in [5] and further studied in [4]. The *neighbourhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an *extreme vertex* if the sub graph induced by its neighbourhood is complete. A vertex v in a connected graph G is a *cut - vertex* of G if $G - v$ is disconnected. A *rooted tree* is a tree with a designated vertex called root, each edge is implicitly directed away from the root. The *closed geodetic numbers* of graphs is studied by Aniversario *et. al* in [2] and extended by Adawiya Amanodin *et. al* in

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[1]. The concept involves closed geodetic closure of a set $S \subset V(G)$ of a graph G denoted by $I[S]$, which is the set of all vertices on a geodesics, and is the shortest path between two vertices in S. This idea evolved from two classes of graph theoretical games called achievement and avoidance games. It is modified for the purpose of closed geodetic concept and the idea goes like this: The first player X take a vertex v_1 of $V(G)$. The second player Y then take $v_2 \neq v_1$ and determine $I[S_2]$ for $S_2 = \{v_1, v_2\}$. If $I[S_2] \neq V(G)$, then X select $v_3 \in I[S_2]$ for $S_3 = \{v_1, v_2, v_3\}$. Similar way X and Y alternately select a new vertex. The first player who select a vertex $v_k \notin I[S_{k-1}]$ such that $I[S_k] = V(G)$ for $S_k = \{v_1, v_2, v_3 \ldots v_k\}$ declare as the winner in the achievement game; in the avoidance game he is the loser. In this paper we extended the idea of connected closed geodetic number of a graph in to connected closed monophonic number of a graph.

Basic concepts and definitions

Definition: For every two vertices u and v of G, the symbol $I[u,v]$ is the interval containing u, v and all vertices lying in some $u - v$ monophonic path. A subset M of $V(G)$ is a *monophonic cover* of G if $I[M] = V(G)$ where $I[M] = \cup_{u,v \in M} I[u,v]$. The set $I[M]$ is called the *monophonic closure* of M in G.

Example: Consider the graph G given in **Fig 1**. Here $M = \{v_1, v_5\}$ is a monophonic cover.

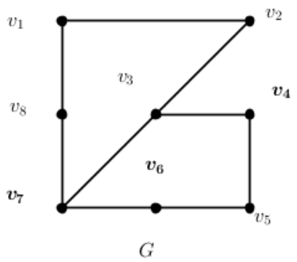

Fig 1

Definition: Let G be a connected graph and $M \subset V(G)$. M is called *closed monophonic cover* of G if $M = \{v_1, v_2, v_3 \ldots v_k\}$ is obtained by choosing the vertices on the following way.

(i) $v_1 \neq v_2$

(ii) $v_i \notin I[M_{i-1}]$ for $3 \leq i \leq k$ and

(iii) $I[M_k] = V(G)$ where $M_i = \{v_1, v_2, v_3 \ldots v_i\}$ for all $i = 1, 2, 3 \ldots k$

The collection of all closed monophonic cover of G is denoted by $C^*_m(G)$. The *closed monophonic number* of G is given by, $cmn(G) = \min\{|M| : M \in C^*_m(G)\}$. A set $M \in C^*_m(G)$ with $|M| = cmn(G)$ is called *minaml closed monophonic set* of G and is denoted by $mcm(G)$.

Example: Consider the graph G given in **Fig 2**. Here $M_1 = \{v_1\}$ is not monophonic cover. $M_2 = \{v_1, v_3\}$ is not a closed monophonic cover, since $v_5 \notin I[M_2]$. $M_3 = \{v_1, v_3, v_5\}$ is a closed monophonic cover. Here $cmn(G) = 3$.

Definition: The *monophonic number*, $mn(G)$ of a graph G is the minimum cardinality among all monophonic covers of G. That is $mn(G) = \min\{|M| : M \subset V(G)\}$ and $I[M] = V(G)$. Also, a monophonic cover of smallest cardinality is called *minimal monophonic set of G*.

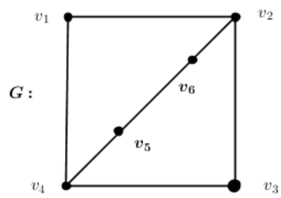

Fig 2

Connected closed monophonic number of a graph

Definition: Let G be a connected graph of order n. Let $M \subset V(G)$ such that $M \in C^*_m(G)$. Then the *connected closed monophonic number* of a graph G is denoted by $ccmn(G)$ and is defined as $ccmn(G) = \min\{|M| : < M > \text{ is connected}\}$ where $< M >$ is the induced sub graph generated by M.

Example: Consider the graph G given in **Fig 3**. Here $M_1 = \{v_1, v_4, v_7\}$ is a closed monophonic cover of G. But $< M_1 >$ not connected. $M_2 = \{v_1, v_2, v_3, v_4, v_6, v_7\}$ is a connected closed monophonic cover of G. That is $cmn(G) = 3$ and $ccmn(G) = 6$. Note that the set $M_2 = \{v_1, v_2, v_5, v_4, v_6, v_7\}$ is also a connected closed monophonic cover of G.

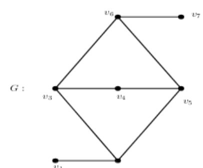

Fig 3

Remark: The following results are immediately. For any connected graph G, $ccmn(G) > cmn(G)$. This follows from the definition of closed monophonic cover and connected closed monophonic cover of a graph. There may be more than one connected closed monophonic cover for a given graph.

Theorem: For any connected non trivial graph G of order $n, 2 \leq cmn(G) \leq ccmn(G) \leq n$.

Proof: Since G is a connected non trivial graph, $|G| \geq 2$.Thus for any connected closed monophonic set need at least two vertices. Therefore $ccmn(G) \geq 2$. Let M be any connected closed monophonic cover of G with minimum cardinality. Then by the remark 3.3, $cmn(G) \leq ccmn(G)$. Also $V(G)$ induces a connected closed monophonic cover of G, we have $ccmn(G) \leq n$.
$2 \leq cmn(G) \leq ccmn(G) \leq n$.

Remark: The bounds in the Theorem (3.4) are sharp. For $G = K_2$, $ccmn(G) = 2$. From the graph in **Fig 3,** $cmn(G) < ccmn(G)$ strictly. Take $G = K_4$.Then $M = \{v_1, v_2, v_3, v_4\}$ is the unique connected closed monophonic cover of G, that is $ccmn(G) = n$.

Theorem: Let $G = P_n$, then $ccmn(G) = n$, for all $n \geq 2$

Proof: Consider the graph G given in **Fig 4**. Let $G = P_n$, the path graph of n vertices. $V(G) = \{v_1, v_2, \ldots v_n\}$ be the set of all vertices in G. Take $M_1 = \{v_1\}$, $M_2 = \{v_1, v_2\}$. Then $I[v_1, v_2] = \{v_1, v_2\} = M_2$. Now $v_3 \notin I[M_2]$. Take $M_3 = M_2 \cup \{v_3\} = \{v_1, v_2, v_3\}$. Clearly $I[M_3] = \{v_1, v_2, v_3\} = M_3$. Thus for any $i = 1, 2, \ldots k < n$; $I[M_i] = M_{i-1} \cup \{v_i\} = M_i$ and $v_i \notin I[M_{i-1}]$. Thus minimum number of vertices in the set M that induces a connected graph for $M \in C^*_m(G)$ if $M = M_{n-1} \cup \{v_n\} = \{v_1, v_2, \ldots v_n\} = V(G)$. Also $< M >$ is a connected closed monophonic cover of G. Thus $ccmn(P_n) = n$ for all $n \geq 2$.

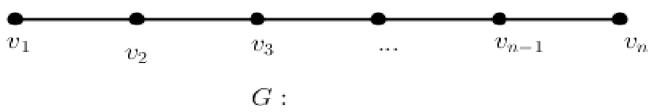

Fig 4

Corollary: If $n = 2$, then $cmn(P_n) = ccmn(P_n)$

Theorem: Let G be the complete graph K_p of order p.Then $ccmn(G) = p$.

Proof: Let $G = K_p$. Then all vertices are mutually adjacent. There for, any $u, v \in V(G)$; $I[u, v] = \{u, v\}$. Now $V(G)$ is the only set of vertices of G such that $I[V(G)] = V(G)$ and $< V(G) >$ is connected. Thus $ccmn(G) = p$.

Corollary: Note that converse of this theorem need not be true. Consider the graph G_2 in figure 05(b). Here $ccmn(G) = 7 = |V(G)|$. But it is not complete.

Illustration: Consider the graph G_1 given in **Fig 5a** and graph G_2 given in **Fig 5b**. Take $M_1 = \{v_1, v_2, v_3\}$ in graph G_1. M_1 is a not connected closed monophonic cover and that $ccmn(G_1) \neq 3$. In G_2, $M_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} = V(G)$ is the unique connected closed monophonic cover.But this graph is not complete

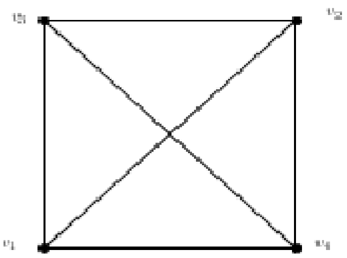

Fig 5a

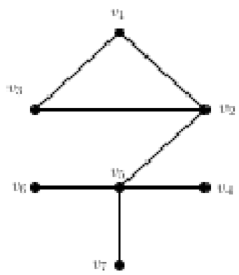

Fig 5b

Theorem: Let G be a connected graph of order p. Then $cmn(G) = ccmn(G)$ if and only if $G = K_p$, complete graph of order $p \geq 2$.

Proof: Let $G = K_p$. Then by theorem (3.3) $cmn(G) = ccmn(G) = p$. Conversely, let $cmn(G) = ccmn(G)$.We have to prove $G = K_p$. On the contrary suppose $G \neq K_p$. Then there exit $u, v \in V(G)$ such that $d(u, v) = 2$. Let u, w, v be a $u - v$ monophonic. Construct $M = \{v_1, v_2 \ldots v_k\} = V(G) - \{w\}$ for which $v_1 = u$ and $v_{k-1} = $ v and $I[M] = V(G)$. Thus M is a closed monophonic cover but $< M >$ is not connected. Therefor $cmn(G) < ccmn(G)$. This is a contradiction. There for $G = K_p$.

Illustration:

Consider the graph G given in figure 06. Take $M = \{v_1, v_3\}$. Then M is closed monophonic cover of G.But M is not connected. Clearly G is not complete.

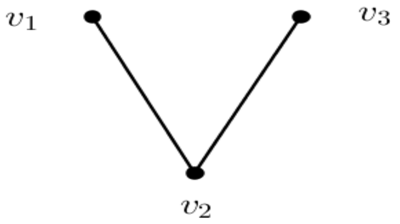

Fig 6

Theorem:

Let $G = K_{m,n}$ complete bipartite graph with partition m, n. Then $ccmn(G) = \min\{m, n\} + 1$; for $m, n \geq 2$. *Proof*: Let $K_{m,n}$ be a bipartite graph and let U, W be the bipartite set of $V(G)$ with $|U| = m$ and $|W| = n$. The only closed monophonic covers of $K_{m;n}$ are $U, W, U\{w\}$ and $W\{u\}$ where $u \in U; w \in W$. Also $U\{w\}$and $W\{u\}$ are the only connected closed monophonic cover of G. There for $ccmn(G) = \min\{|U| + 1; |W| + 1\} = \min\{m + 1; n + 1\} = \min\{m, n\} + 1.$

Illustration:

Consider the graph G given in **Fig 7**. Take $U = f u_1, u_2, u_3\}, W = \{w_1, w_2\}$. Then $M_1 = U$; $M_2 = W, M_3 = \{w_1, u_1, w_2\}$, *etc*. $M_4 = \{w_1, u_2, w_2\}$ *etc*. are monophonic covers of G. Here M_1, M_2 are not connected closed monophonic covers. But M_3, M_4 are connected. Clearly $ccmn(G) = |M_3| = 3 = \min\{2, 3\} + 1$.

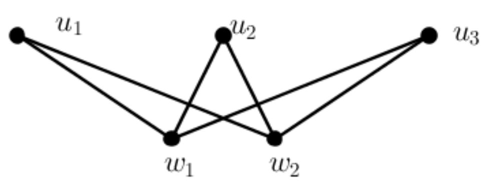

Fig 7

Corollary:

If $m = n$ then $ccmn(G) = n + 1$

Theorem:

Let $G = C_n$. Then for $n \geq 3, ccmn(G) = 3$

Proof: Let $G = C_n$ be the cycle of length n and $V(G) = \{v_1, v_2 \ldots v_n\}$ be the set of all vertices of G. If $n = 3$, then $C_n = K_3$ and by theorem 3.8 $ccmn(G) = 3$.Let $n > 3$. Take $M_1 = \{v_1\}$, $M_2 = \{v_1, v_2\}$ and $M_2 = \{v_1, v_2, v_3\}$. Then M_3 is a connected closed monophonic path in C_n Thus $ccmn(C_n) = 3$.

Connected closed monophonic number of a tree

Theorem: Each extreme vertex of a connected graph G belongs to every connected closed monophonic cover of G.

Proof: By theorem 2.3 of [5], each extreme vertex of a connected graph belongs to every monophonic set. Since connected closed monophonic cover of a graph is also a monophonic set, the result follows.

Theorem: Let G be a connected graph and v be a cut vertex. Let M be a connected closed monophonic cover of G. Then every component of $G - v$ contains at least one element of M.

Proof: By theorem [5], if M is a monophonic set of a connected graph G and v be a cut vertex of G, then every component of $G - v$ contains an element of M. Since every connected closed monophonic cover of G is also a monophonic set, the result follows.

Theorem: Let G be a connected graph and M be a connected closed monophonic cover of G. Then every cut vertex of G lies in M.

Proof: Let M be a connected closed monophonic cover of G and let $v \in V(G)$ be a cut vertex. Let $G_1, G_2 \ldots G_k (k \geq 2)$ be the components of $G - v$. By theorem (4.2) M contains at least one vertex from each G_i for $1 \leq i \leq k$. Since the sub graph induced by M is connected, it follows that $v \in M$. Thus every cut vertex belongs to M.

Corollary: For any connected graph G with k extreme vertices and l cut vertices, $ccmn(G) \geq max\{2, k + l\}$.

Proof: From the theorem 3.1 we have $ccmn(G) \geq 2$. Also for any connected graph G set of cut vertices and set of extreme vertices are disjoint. From theorems 4.1 and 4.3, every extreme vertices and cut vertices lies in the connected closed monophonic set. There for $ccmn(G) \geq max\{2, k + l\}$.

Theorem: For any non-trivial tree T of order n, $ccmn(T) = n$.

Proof: For trivial tree the result is true. Let $n \geq 2$. First, $ccmn(T) \leq n$. Also from corollary 4.4 $ccmn(T) \geq \max\{2; k + l\}$ where k is the set of extreme vertices of T and l is the set of all cut vertices. Since every vertex of a non-trivial tree T is either a cut vertex or extreme vertex, $l + k = n$. There for, $ccmn(T) \geq \max\{2, n\}$. That is, $ccmn(T) \geq n$. Hence, $n \leq ccmn(T) \leq n$ and the result follows.

Illustration:

Consider a rooted tree T given in **Fig 8**. Here $U = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ are extreme vertices and $W = \{v_7, v_8, v_9, v_{10}\}$ are cut vertices. Clearly connected closed monophonic cover M of T contains all the vertices. That is $|M| = |T| = 10$

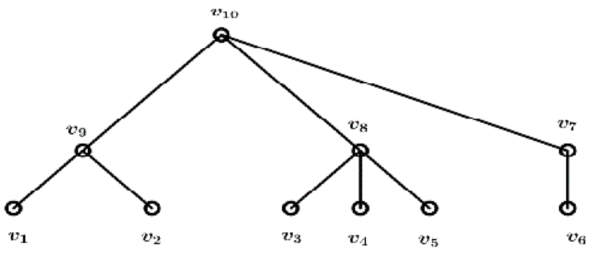

Fig 8

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